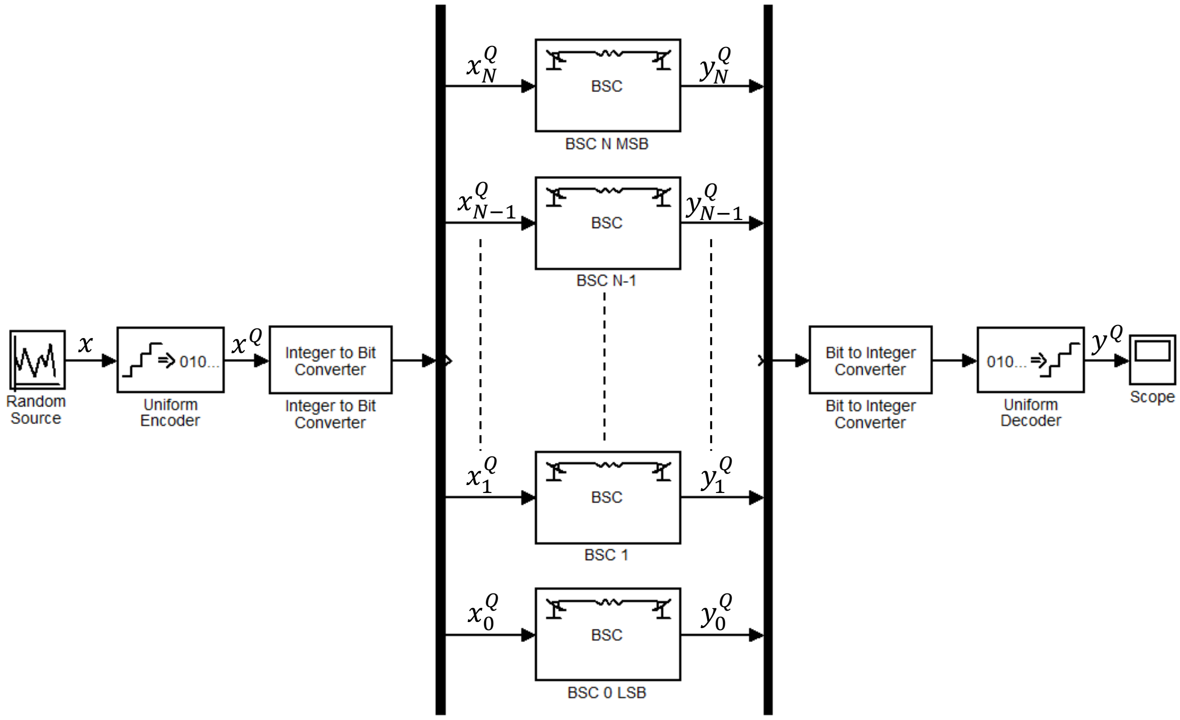
Does Information always represent quantity meant?

[Motivated from Xin Li, "Maximum-information storage system: concept, implementation and application," IEEE/ACM International Conference on Computer-Aided Design (ICCAD), pp. 39-46, 2010.]

Problem

In many scenarios involving transmission/storage of real valued data, the aim is to minimize distortion in a uniformly distributed continuous signal over $[-1,1]$ by passing through $N+1$ Binary Symmetric Channels having error probability $\{\alpha_i\}_{i=0}^{N}$ subject to the constraint that $\prod_0^N \alpha_n = k^{N+1}c$ with each $\alpha_n \leq \alpha_{max}$. The design parameters are N and the error probabilities $\{\alpha_i\}_{i=0}^{N}$. This can be represented by following schematic:



One important such scenario is SRAM design for signal processing applications, where design goal is to store maximum data reliably in a given amount of area. Assume that $N+1$ SRAM cells are used to store the $N+1$ bits $\{y_n^Q\}_{n=0}^{N}$ representing the real valued data x and the silicon area of these memory cells is denoted as $\{s_n\}_{n=0}^{N}$. The total silicon area s_{Total} is simply the summation of individual memory cell:

$$s_{Total} = \sum_{n=0}^{N} s_n$$

For a given CMOS process, the relation between failure probability α_n and cell area s_n can be approximated as

$$\alpha_n = ke^{-\beta s_n}$$

This translates above mentioned area constraint as

$$\prod_{n=0}^{N} \alpha_n = k^{N+1} e^{-\beta s_{Total}}$$

Also in a given CMOS process the size of memory cell can be no smaller than a certain value, say s_{min}, which means $\forall n\ s_n \geq s_{min}$ or equivalently $\forall n\ \alpha_n \leq \alpha_{max}$. So this problem beautifully fits the general problem being targeted.

Mutual Information Maximization

Let us first evaluate sort of Channel Capacity measure of such system, i.e.

$$C_N = \max_{p(y^Q|x)} I(x; y^Q)$$
$$= \max_{p(y^Q|x)} H(y^Q) - H(y^Q|x)$$

Next observe that whether we have knowledge of x accurately or just have knowledge of x^Q, it does not makes difference in our uncertainty about y^Q, i.e. $H(y^Q|x) = H(y^Q|x^Q)$. In fact one can claim that $p(y^Q|x) = p(y^Q|x^Q)$. Further as it is given that $x \sim U\left(0, \frac{1}{3}\right)$, one can easily check that $y^Q \sim U\left(0, \frac{1}{3}\right)$. So, the problem reduces to

$$C_N = (N+1) - \min_{p(y^Q|x^Q)} H(y^Q|x^Q)$$

As $\left(x_0^Q, x_1^Q, \dots, x_N^Q\right) = f(x^Q)$ and $y^Q = f^{-1}\left(y_0^Q, y_1^Q, \dots, y_N^Q\right)$, where $f: dom(x^Q) \to \{0,1\}^{N+1}$ is a bijection , it implies $H(y^Q|x^Q) = H\left(y_0^Q, y_1^Q, \dots, y_N^Q \middle| x_0^Q, x_1^Q, \dots, x_N^Q\right)$. One can easily show that each of y_n^Q are independent of each other so that

$$H\left(y_0^Q, y_1^Q, \dots, y_N^Q \middle| x_0^Q, x_1^Q, \dots, x_N^Q\right) = \sum_{n=0}^{N} H\left(y_n^Q \middle| x_0^Q, x_1^Q, \dots, x_N^Q\right)$$

For Binary Symmetric Channel $H\left(y_n^Q \middle| x_0^Q, x_1^Q, \dots, x_N^Q\right)$ is nothing but $H(\alpha_n)$, so

$$C_N = (N+1) - \min_{\alpha_0, \alpha_1, \dots, \alpha_N} \sum_{n=0}^{N} H(\alpha_n)$$

This can be formulated as an optimization problem as

$$\begin{aligned} &\text{minimize} && \sum_{n=0}^{N} H(\alpha_n) \\ &\text{subject to} && \prod_{n=0}^{N} \alpha_n = k^{N+1} c \\ & && \forall n \;\; \alpha_n \le \alpha_{max} \end{aligned}$$

Applying method of Lagrange multipliers

$$\Lambda(\alpha_1, \alpha_2, \dots, \alpha_N, \lambda) = \sum_{n=0}^{N} H(\alpha_n) + \lambda \left(\prod_{n=0}^{N} \alpha_n - k^{N+1} c \right)$$

Location of optimum value is found out by solving the set of $N+2$ equations: $\nabla\Lambda(\alpha_1, \alpha_2, \dots, \alpha_N, \lambda) = 0$

Consider the n^{th} equation

$$-\log_2\left(\frac{\alpha_n^*}{1-\alpha_n^*}\right) + \lambda \frac{k^{N+1} c}{\alpha_n^*} = 0$$
$$\lambda = \frac{\alpha_n^*}{k^{N+1} c} \log_2\left(\frac{\alpha_n^*}{1-\alpha_n^*}\right)$$

This implies that $\alpha_0^* = \alpha_1^* = \cdots = \alpha_N^*$, and using $(N+2)^{th}$ equation

$$\alpha_0^* = \alpha_1^* = \cdots = \alpha_N^* = k c^{\frac{1}{N+1}}$$

Finally, the channel capacity of the system is, when $k c^{\frac{1}{N+1}} \le \alpha_{max}$

$$C_N = (N+1)\left(1 - H\left(k c^{\frac{1}{N+1}}\right)\right)$$

If $k c^{\frac{1}{N+1}} > \alpha_{max}$, then we need to resort to computational methods.

The optimization with respect to N can be carried out by simply evaluating C_N over allowed values of N.

Minimum Distortion

Since it is known that $x \sim U\left(0, \frac{1}{3}\right)$, uniform quantizer is the best bet. The quantizer output can be decomposed bit-wise in a 2's complement fashion as

$$x^Q = -x_0^Q + \sum_{n=1}^{N} x_n^Q 2^{-n}$$

As it is given that $x \sim U\left(0, \frac{1}{3}\right)$, we can easily observe that $x^Q \sim U\left(-\frac{1}{2}2^{-N}, \frac{1}{3}(1 - 4^{-N})\right)$ and each $x_n^Q \sim B(0.5)$ i.i.d. Similarly after the Binary Symmetric Channel, $x_n^Q \sim B(0.5)$ results in each $y_n^Q \sim B(0.5)$ and thus $y^Q \sim U\left(-\frac{1}{2}2^{-N}, \frac{1}{3}(1 - 4^{-N})\right)$.

If the distortion is taken in mean square error (MSE) sense, the given task reduces to well-known minimum mean square error (MMSE) problem:

$$\begin{aligned} D_N &= \max_{p(y^Q|x)} E[(x - y^Q)^2] \\ &= \max_{p(y^Q|x)} E[x^2 + (y^Q)^2 - 2xy^Q] \\ &= E[x^2] + E[(y^Q)^2] - 2 \min_{p(y^Q|x)} E[xy^Q] \\ &= \frac{1}{3} + \frac{1}{3} - \frac{1}{12}4^{-N} - 2 \min_{\alpha_0, \alpha_1, \dots, \alpha_N} \left(\frac{1}{3} - \frac{1}{12}4^{-N} - \frac{1}{2}\sum_{n=0}^{N} \alpha_n 4^{-n}\right) \\ &= \min_{\alpha_0, \alpha_1, \dots, \alpha_N} \left(\frac{1}{12}4^{-N} + \frac{1}{2}\sum_{n=0}^{N} \alpha_n 4^{-n}\right) \end{aligned}$$

This can be formulated as an optimization problem as

$$\begin{aligned} \text{minimize} \quad & \frac{1}{12}4^{-N} + \frac{1}{2}\sum_{n=0}^{N} \alpha_n 4^{-n} \\ \text{subject to} \quad & \prod_{n=0}^{N} \alpha_n = k^{N+1}c \\ & \forall n \;\; \alpha_n \le \alpha_{max} \end{aligned}$$

Applying method of Lagrange multipliers

$$\Lambda(\alpha_1, \alpha_2, \dots, \alpha_N, \lambda) = \frac{1}{12}4^{-N} + \frac{1}{2}\sum_{n=0}^{N} \alpha_n 4^{-n} - \lambda\left(\prod_{n=0}^{N} \alpha_n - k^{N+1}c\right)$$

Location of optimum value is found out by solving the set of $N + 2$ equations: $\nabla\Lambda(\alpha_1, \alpha_2, \dots, \alpha_N, \lambda) = 0$

Consider the n^{th} equation

$$4^{-n} - \lambda \frac{k^{N+1}c}{\alpha_n^*} = 0$$

$$\alpha_n^* = (\lambda k^{N+1}c)4^n$$

using $(N + 2)^{th}$ equation

$$k^{N+1}c = (\lambda k^{N+1}c)^{N+1} 2^{N(N+1)}$$

$$\lambda k^{N+1}c = \frac{1}{2^N} kc^{\frac{1}{N+1}}$$

$$\alpha_n^* = \frac{1}{2^{N-2n}} kc^{\frac{1}{N+1}}$$

which is valid if $2^{N(N+1)}k^{N+1}c \le \alpha_{max}$, otherwise computational method has to be employed.

Simulation Result

Chosen parameters:

$$k = 1.5 \times 10^{-4}$$
$$c = 1.406861712446147 \times 10^{-16}$$
$$\alpha_{max} = 7.228634851353038 \times 10^{-5}$$

Results:

	Distortion Minimization	Corresponding Info Max
Number of Bits	12	12
Mutual Information	11.9929	11.9984
Distortion	1.7649×10^{-7}	9.5709×10^{-6}
Distortion (Simulation)	1.5858×10^{-7}	8.2366×10^{-6}
Bit Error Rate (Simulation)	3.6800×10^{-5}	5.8000×10^{-6}

Commentary

Simply maximizing mutual information does not result in minimum distortion always. They are same only when each bit has equal importance, which is not the case in this scenario.

Appendix

$$\begin{aligned}
E[x^Q y^Q] &= E\left[\left(-x_0^Q + \sum_{m=1}^{N} x_m^Q 2^{-m}\right)\left(-y_0^Q + \sum_{n=1}^{N} y_n^Q 2^{-n}\right)\right] \\
&= E\left[x_0^Q y_0^Q - \sum_{n=1}^{N} x_0^Q y_n^Q 2^{-n} - \sum_{m=1}^{N} x_m^Q y_0^Q 2^{-m} + \sum_{m=1}^{N}\sum_{n=1}^{N} x_m^Q y_n^Q 2^{-(m+n)}\right] \\
&= E[x_0^Q y_0^Q] - \sum_{n=1}^{N} E[x_0^Q y_n^Q] 2^{-n} - \sum_{m=1}^{N} E[x_m^Q y_0^Q] 2^{-m} + \sum_{m=1}^{N}\sum_{n=1}^{N} E[x_m^Q y_n^Q] 2^{-(m+n)} \\
&= \frac{1-\alpha_0}{2} - \frac{1}{4}(1-2^{-N}) - \frac{1}{4}(1-2^{-N}) + \sum_{m=1}^{N}\sum_{n=1}^{N}\left\{\left(\frac{1}{4} - \frac{\alpha_m}{2}\right)\delta[m-n] + \frac{1}{4}\right\} 2^{-(m+n)} \\
&= \frac{1-\alpha_0}{2} - \frac{1}{2}(1-2^{-N}) + \frac{1}{4}\sum_{n=1}^{N} 2^{-2n} - \frac{1}{2}\sum_{n=1}^{N} \alpha_n 2^{-2n} + \frac{1}{4}\sum_{m=1}^{N}\sum_{n=1}^{N} 2^{-(m+n)} \\
&= \frac{1-\alpha_0}{2} - \frac{1}{2}(1-2^{-N}) + \frac{1}{12}(1-4^{-N}) - \frac{1}{2}\sum_{n=1}^{N} \alpha_n 4^{-n} + \frac{1}{4}(1-2^{-N})^2 \\
&= \frac{1}{3} + \frac{1}{6}4^{-N} - \frac{1}{2}\sum_{n=1}^{N} \alpha_n 4^{-n}
\end{aligned}$$

$$\begin{aligned}
E[xy^Q] &= \sum_{y^Q} \int_{-1}^{1} x y^Q f(x, y^Q) dx \\
&= \sum_{y^Q} \sum_{i=-2^N}^{2^N-1} y^Q \int_{i/2^N}^{(i+1)/2^N} x f(x, y^Q) dx
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{y^Q} \sum_{i=-2^N}^{2^N-1} y^Q \int_{i/2^N}^{(i+1)/2^N} x f(y^Q|x) f(x) dx \\
&= \sum_{y^Q} \sum_{i=-2^N}^{2^N-1} y^Q p\left(y^Q| \frac{i}{2^N}\right) \int_{i/2^N}^{(i+1)/2^N} x f(x) dx \\
&= \sum_{y^Q} \sum_{i=-2^N}^{2^N-1} y^Q p\left(y^Q| \frac{i}{2^N}\right) \int_{i/2^N}^{(i+1)/2^N} \frac{x}{2} dx \\
&= \sum_{y^Q} \sum_{i=-2^N}^{2^N-1} y^Q p\left(y^Q| \frac{i}{2^N}\right) \frac{1}{2^{N+1}} \left(\frac{i}{2^N} + \frac{1}{2^{N+1}}\right) \\
&= \sum_{y^Q} \sum_{x^Q} y^Q p(y^Q|x^Q) \frac{1}{2^{N+1}} \left(x^Q + \frac{1}{2^{N+1}}\right) \\
&= \sum_{y^Q} \sum_{x^Q} y^Q p(y^Q|x^Q) p(x^Q) \left(x^Q + \frac{1}{2^{N+1}}\right) \\
&= \sum_{y^Q} \sum_{x^Q} x^Q y^Q p(x^Q, y^Q) + \frac{1}{2^{N+1}} \sum_{y^Q} \sum_{x^Q} y^Q p(x^Q, y^Q) \\
&= E[x^Q y^Q] + \frac{1}{2^{N+1}} E[y^Q] \\
&= E[x^Q y^Q] - \frac{1}{4} 4^{-N} \\
&= \frac{1}{3} - \frac{1}{12} 4^{-N} - \frac{1}{2} \sum_{n=1}^{N} \alpha_n 4^{-n}
\end{aligned}
$$